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Three notes on coprime packedness

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Abstract

We characterise coprimely packed rings ideal theoretically and relate the concept of coprimely packedness of R to that of $R(X)$, where R is a ring and X an indeterminate over R. © 2000 Elsevier Science B.V. All rights reserved.

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0. Introduction

Let R be a commutative ring with identity and I an ideal of R. Then I is said to be coprimely packed by prime ideals of R if whenever I is coprime to each element of a family of prime ideals of R, I is not contained in the union of prime ideals in the family. If every ideal of R is coprimely packed, then we say R is coprimely packed.

It is shown that in an integral domain R, $MaxSpecR$ is comprimely packed if and only if for each maximal ideal M of R there is a principal ideal I of R contained in M such that $(R/I)_M \simeq R/I$, and $(R/I)_N = 0$, for all $N \in MaxSpecR - \{M\}$. Let R be an almost Dedekind domain. Then $MaxSpecR$ is coprimely packed if and only if R is a Dedekind domain with torsion ideal class group. We then show that in a finite-dimensional QR-domain R, $MaxSpecR$ is comprimely packed if and only if every maximal ideal of R is the radical of a principal ideal. Finally, we show that if R is coprimely packed then so is $R(X)$, and point out that the coprimely packedness of $R(X)$ does not imply the same for R.

Throughout, R will denote a commutative ring with identity, $SpecR$ will denote the set of prime ideals of R, and $MaxSpecR$ the set of maximal ideals of R. For any maximal ideal M of R, $\Omega(M)$ will denote the set $MaxSpecR - \{M\}$ and S_M will

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denote the set $R - \bigcup_{N \in \Omega(M)} N$. If R is an integral domain, K will denote the field fractions of R.

1. Integral domains with coprimely packed prime ideals

Let R be a ring and $SpecR$ be the set of prime ideals of R. Then we say $SpecR$ is coprimely packed if whenever a prime ideal P of R is coprime to each element of a family of prime ideals of R, P is not contained in the union of the prime ideals in the family. We also say that $MaxSpecR$ is coprimely packed if each maximal ideal of R is not contained in the union of the other maximal ideals.

Here we characterise integral domains with coprimely packed set of prime ideals. We begin with the following preliminary lemma which will be needed in the proof of the next statement.

Lemma 1.1. *Let $M \in MaxSpecR$, and let $A \neq 0$ be an R-module with the property that $A_N = 0$ for all $N \in \Omega(M)$. Then the map $a \to a/1$ gives an isomorphism from A* to A_M.

Proof. Let $0 \neq a \in A$, and let $I = Ann_R\, a$. Suppose $N \in \Omega(M)$. Since $A_N = 0$, we see that $a/1$ is zero in A_N. Thus there is an $s \in R - N$ with $sa = 0$, so that $s \in I$. This shows that $I \nsubseteq N$. Since $a \neq 0$, $I \neq R$. It follows that I is contained in M, but is not contained in any $N \in \Omega(M)$.

We will now show that the map $a \to a/1 \in A_M$ is a monomorphism. Suppose $a \neq 0$ in A. Then we already know $I \subseteq M$. This shows that $a/1$ is not zero in A_M.

To see that our map is onto, pick any $a/t \in A_M$ (so $a \in A$ and $t \in R - M$). If $a = 0$, then $0 \to 0/1 = a/t$ and we are done. Thus suppose $a \neq 0$, so that $I = Ann_R\, a$ is contained in M but is not contained in any $N \in \Omega(M)$. Then $Rt + I = R$. Write $1 = rt + i$ with $r \in R$ and $i \in I$. Multiplying by a, we have $a = atr + ai = atr + 0 = atr$. Now, $ar \to ar/1 = atr/t = a/t$, and so our map is onto. $\square$

Theorem 1.2. *For an integral domain R the following statements are equivalent*:

(i) *$MaxSpecR$ is coprimely packed.*

(ii) *For any maximal ideal M of R, there is a principal ideal I of R contained in M such that if $A = R/I$, then $A_N = 0$, for all $N \in \Omega(M)$.*

(iii) *$S^{-1}R \nsubseteq R_M$, where S is the complement in R of the union of elements in any non-empty subset X of $MaxSpecR$ and $M \in MaxSpecR - X$.*

Proof. (i)→(ii). Let M be any maximal ideal of R. Then $M \nsubseteq \bigcup_{N \in \Omega(M)} N$.

Therefore, it follows that there is an element $x \in M$ which is not contained in any other maximal ideal of R. Put $I = Rx$ and $A = R/I$. But then by Lemma 1.1, we have $A_N = 0$, for all $N \in \Omega(M)$.

(ii)→(iii). Suppose that there exists a non-empty subset X of $MaxSpecR$ and a maximal ideal M of R not in X such that $S^{-1}R \subseteq R_M$, $S = R - \bigcup_{N \in X} N$. Then it follows that $M \subseteq \bigcup_{N\in X} N$. But this is a contradiction to the fact that there is a principal ideal I of R contained in M such that $(R/I)_N = 0$, for all $N \in X \subseteq \Omega(M)$. Therefore $S^{-1}R \nsubseteq R_M$.

(iii)→(i). Straightforward. □

Theorem 1.3. *For an integral domain R the following statements are equivalent*:

(i) *For each maximal ideal M of R. $S_M^{-1}R \otimes_R R_M = K$.*

(ii) *SpecR is comprimely packed and every non-zero prime ideal of R is contained in only one maximal ideal of R.*

(iii) *$Hom_R(K/R_P, K/R_Q) = 0$, for each prime ideal P contained in a given maximal ideal M of R and each prime ideal Q contained in $\bigcup_{N\in\Omega(M)} N$.*

Proof. (i)→(ii). Let P be any non-zero prime ideal of R and suppose that there are distinct maximal ideals M_1, M_2 of R such that $P \subseteq M_1$ and $P \subseteq M_2$. Then it follows that $(P_{M_1})_{M_2}$ is a proper prime ideal of $R_{M_1} \otimes_R R_{M_2}$. But this is impossible, since $R_{M_1} \otimes_R R_{M_2} = K$. Therefore, it follows that every non-zero prime ideal of R is contained in only one maximal ideal of R. We next show that if P is any non-zero prime ideal of R and X any non-empty subset of *SpecR* such that $P + Q = R$, for all $Q \in X$, then $P \nsubseteq \bigcup_{Q\in X} Q$. Let M be the only maximal ideal of R containing P and $\Gamma(P)$ be the set of prime ideals of R not contained in M and set $S_X = R - \bigcup_{Q\in X} Q$, $S_P = R - \bigcup_{Q\in\Gamma(P)} Q$. Then clearly $X \subseteq \Gamma(P)$, $\Omega(M) \subseteq \Gamma(P)$ and $S_P = S_M \subseteq S_X$. Hence, $K = S_P^{-1}R \otimes_R R_M \subseteq S_X^{-1}R \otimes_R R_P$, and so $P \nsubseteq \bigcup_{Q\in X} Q$. Therefore, *SpecR* is coprimely packed.

(ii)→(iii). Let $f : K/R_P \to K/R_Q$ be a homomorphism of R-modules. Then for any maximal ideal W of R, $f_W : (K/R_P) \otimes_R R_W \to (K/R_Q) \otimes_R R_W$ is an R_W-homomorphism of R_W-modules. Since $P \subseteq M$ and $Q \subseteq \bigcup_{N\in\Omega(M)} N$, it follows that no maximal ideal of R contain both P and Q. Therefore, we either have $(K/R_P) \otimes_R R_W = 0$ or $(K/R_Q) \otimes_R R_W = 0$, and so $f_W = 0$. That is $f_W = 0$, for all maximal ideals W of R, and hence $f = 0$.

(iii)→(i). Suppose that there is a maximal ideal M of R such that $R_M \otimes_R S_M^{-1}R \neq K$. Then $R_M \otimes_R S_M^{-1}R$ has a maximal ideal M', and $P = M' \cap R$ is a prime ideal of R contained in M and in $\bigcup_{N\in\Omega(M)} N$. But then the identity mapping $f : K/R_P \to K/R_P$ is a non-zero homomorphism, a contradiction. Therefore we must have $R_M \otimes_R S_M^{-1}R = K$. □

We now would like to point out that in the statements of Proposition 1.3 and Theorem 1.4 of [1], the expression $R_Q \otimes_R (\bigcap_{P\in X} R_P) = K$ should have been as $R_Q \otimes_R S_X^{-1}R = K$, where $S_X = R - \bigcup_{P\in X} P$ and also if R is a GCD domain, then the expressions $\bigcap_{P\in X} R_P$ and $S_X^{-1}R$ are one and the same.

2. Prime avoidance of maximal ideals in Prüfer rings

Recall that R is a Prüfer ring if for each maximal ideal M of R, the local ring R_M is a valuation ring. In this section we characterise coprimely packed Prüfer rings.

Theorem 2.1. *Let R be an almost Dedekind domain (i.e. for each maximal ideal M of R, R_M is a Dedekind domain). Then MaxSpecR is coprimely packed if and only if R is a Dedekind domain with torsion ideal class group.*

Proof. Since the if part is clear, we prove the only if part. Let M be any maximal ideal of R and $x \in M - \bigcup_{N\in\Omega(M)} N$. Then $\sqrt{x} = M$. Hence, $\sqrt{R(x)_M} = M_M$ in R_M. But R_M is a discrete valuation ring. Therefore $(Rx)_M$ is M_M-primary and so $(Rx)_M = M_M^n$ for some positive integer n. Since M is maximal it follows that $Rx = M^n$ in R. Hence M is invertible in R. Therefore R is a Dedekind domain with torsion ideal class group. □

Proposition 2.2. *For a Prüfer domain R the following statements are equivalent:*
(i) *MaxSpecR is coprimely packed.*
(ii) *Each maximal ideal M of R contains a principal ideal I such that $\sqrt{I}$ is a prime ideal of R contained only in M.*

Proof. (i)→(ii). Let M be any maximal ideal of R and x an element in M which is not contained in any other maximal ideal of R. Since $\sqrt{Rx}$ is the intersection of all the prime ideals of R containing x, it follows that any prime ideal of R containing x is contained only in the maximal ideal M. But R is Prüfer. Therefore, the set of prime ideals of R contained in M is linearly ordered, and so $\sqrt{Rx}$ is a prime ideal of R contained only in M.

(ii)→(i). Let X be a non-empty subset of *MaxSpecR* and $M \in$ *MaxSpecR*$-X$. Since by assumption M contains a prime ideal of the form $\sqrt{Rx}$ and $\sqrt{Rx} + N = R$, for all $N \in X$, it follows that $\sqrt{Rx} \nsubseteq \bigcup_{N\in X} N$, and so $M \nsubseteq \bigcup_{N\in X} N$. Therefore *MaxSpecR* is coprimely packed. □

We recall from [3] that an integral domain R is said to be QR-domain if every ring between R and its quotient field K is a quotient ring of R. In [6] such domains are characterised as Prüfer domains with the additional property that the radical of each finitely generated ideal is the radical of a principal ideal. Clearly Bezout domains are QR-domains. On the other hand it is not the case that every QR-domain is a Bezout domain. For example, the QR-domain of Krull dimension one constructed in [4, pp. 139–142] is not Bezout.

Theorem 2.3. *In a finite-dimensional QR-domain R, MaxSpecR is comprimely packed if and only if every maximal ideal of R is the radical of a principal ideal.*

Proof. Let M be any maximal ideal of R and $x \in M - \bigcup_{N\in\Omega(M)} N$. Then by Proposition 2.2, $\sqrt{Rx}$ is a prime ideal of R contained only in M. If $M = \sqrt{Rx}$, then we stop here. If not, then there exists a y in $M - \sqrt{Rx}$, and $\sqrt{Rx + Ry} = \sqrt{Rz}$ for some $z \in M$ (since R is a QR-domain). Clearly, $\sqrt{Rx} \subset \sqrt{Rz}$, and for the same reason as above we have $\sqrt{Rz}$ is a prime ideal contained in M which is not contained in any other maximal ideal of R. Continuing in this way (after a finite number of steps) we can find an element $m \in M - \bigcup_{N \in \Omega(M)} N$ such that $\sqrt{Rm} = M$.

The converse requires no comment. □

Corollary 1. *In a finite-dimensional Bezout domain MaxSpecR is coprimely packed if and only if every maximal ideal is the radical of a principal ideal.*

Corollary 2. *Let R be a semilocal Prüfer domain of finite dimension. Then every maximal ideal of R is the radical of a principal ideal.*

Proof. This follows from the fact that a semilocal Prüfer domain is a Bezout domain. □

Let R be a finite-dimensional Prüfer domain with coprimely packed set of maximal ideals. Then each maximal ideal of R is the radical of an ideal generated by two elements. This suggests that a Prüfer domain (of finite dimension) with coprimely packed set of maximal ideals need not be a *QR*-domain. We now show that under a suitable condition the set of maximal ideals of a *QR*-domain is coprimely packed.

Proposition 2.4. *Let R be a finite-dimensional QR-domain in which every non-zero ideal is contained in only finitely many maximal ideals. Then MaxSpecR is coprimely packed.*

Proof. Let M be any maximal ideal of R and x be a non-zero element of M. Let $M_1, M_2, \ldots, M_n$ be the other maximal ideals of R containing x, and y be an element of M not contained in any one of M_i $(1 \leq i \leq n)$. Then the ideal $Rx + Ry$ of R is contained only in M but not in any other maximal ideal of R. Since R is a *QR*-domain, $\sqrt{Rx + Ry} = \sqrt{Rz}$ and $\sqrt{Rz}$ is a prime ideal of R contained only in M. But then $M = \sqrt{Rm}$, for some $m \in M$ follows from the proof of Theorem 2.3. Therefore *MaxSpecR* is coprimely packed. □

3. Polynomial rings

Let R be a ring and X a transcendental over R. Let S be the set of polynomials $f \in R[X]$ whose coefficients generate the unit ideal R in R. Then S is a multiplicatively closed subset of $R[X]$ and contains no zero divisor. The ring $S^{-1}R[X]$ is denoted by $R(X)$. Identifying R and $R[X]$ with their isomorphic images in $R(X)$, we have $R \subseteq R[X] \subseteq R(X)$ [5, pp. 17–19]. Here we relate the concept of coprimely packedness of R, and $R(X)$.

Theorem 3.1. *If R is coprimely packed, then so is $R(X)$.*

Proof. To prove that $R(X)$ is coprimely packed it is enough to show that every prime ideal of $R(X)$ is coprimely packed by the set of maximal ideals of R [2, Proposition 1.2].

Let P' be any prime ideal of $R(X)$ and $\wedge'$ be any non-empty subset of $MaxSpecR(X)$ such that $P' + M' = R(X)$, for all $M' \in \wedge'$. Since every ideal of $R(X)$ is an extended ideal of $R[X]$, we have $P' = S^{-1}P$ and each $M' \in \wedge'$, $M' = S^{-1}(M[X])$ where P is a prime ideal of $R[X]$ and M is a maximal ideal of R (see, [5, p. 18]). Thus, we have $S^{-1}P + S^{-1}(M[X]) = S^{-1}(P + M[X]) = R(X)$, for all $M \in \wedge$, where $\wedge = \{M \mid S^{-1}(M[X]) \in \wedge'\} \subseteq MaxSpecR$. Hence $(P + M[X]) \cap S \neq \Phi$, for all $M \in \wedge$. Let A be the ideal of R consisting of the coefficients of the polynomials in P. If $A \subseteq M$ for some $M \in \wedge$, then $P \subseteq M[X]$ which contradicts $S^{-1}P + S^{-1}(M[X]) = R(X)$. Thus (since R is coprimely packed) there is a $c \in A$ with $c \notin M$ for all $M \in \wedge$. Let $f(x)$ be a polynomial in P having c as some coefficient. Since $f(x) \notin \bigcup_{M \in \wedge} M[X]$ in $R[X]$, we see that $P' \nsubseteq \bigcup_{M' \in \wedge'} M'$, and so $R(X)$ is coprimely packed. $\square$

Let R be a Dedekind domain whose class group is not torsion. Then $R(X)$ is a principal ideal domain and so is coprimely packed without R being coprimely packed. Therefore the converse of Theorem 3.1 is not true. In [2], it is shown that $Z[X]$ the ring of polynomials over the integers is not coprimely packed. Therefore, in general, the coprimely packedness of R does not imply the same for $R[X]$.

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